New CQ-free optimality criterion for convex SIP problems with polyhedral index sets

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Consider a convex Semi-Infinite Programming (SIP) problem in the form

$$(P): \qquad \min_{x\in\mathbb{R}^n} c(x), \tag{1}$$

$$\text{s.t.} \quad f(x,t) \le 0 \ \ \forall\, t \in T = \{t \in \mathbb{R}^s:\ h_k^T t \le \Delta h_k,\ k \in K\}, \tag{2}$$

where the objective function $c(x), x \in \mathbb{R}^n$, is convex, the constraint function $f(x,t), x \in \mathbb{R}^n, t \in T$, is linear w.r.t. x; $h_k \in \mathbb{R}^s$, $\Delta h_k \in \mathbb{T}$, $k \in K, |K| < \infty$. Notice that the index set T in (P) is a convex polyhedron.

Let X be the feasible set of problem (P): $X = \{x \in \mathbb{R}^n : f(x,t) \le 0, \forall t \in T\}$. Suppose that $f(x,t)$ is sufficiently smooth w.r.t. t for all $x \in X$ and $t \in T$.

Given $t \in T$, denote by $K_a(t) \subset K$ the set of active indices in t, $K_a(t) := \{k \in K:\ h_k^T t = \Delta h_k\}$, and by $L(t)$ the set of feasible directions in T starting from t,

$$L(t) := \{l \in \mathbb{R}^s:\ h_k^T l \le 0,\ k \in K_a(t)\}. \tag{3}$$

Given $x \in X$, the set of active indices in x is $T_a(x) := \{t \in T : f(x,t) = 0\}$.

D e f i n i t i o n 1. Let us say that an index $\bar t \in T$ is immobile in problem (P), if $f(x,\bar t) = 0$ for all $x \in X$.

Denote by T^* the *set of all immobile indices* in problem (P). It is evident that $T^* \subset T_a(x)$ for all $x \in X$.

D e f i n i t i o n 2. The constraints of problem (P) satisfy *the Slater condition* if there exists $\bar x \in X$ such that $f(\bar x, t) < 0, \forall t \in T$.

In [4], it is proved that a convex SIP problem with $X \ne \emptyset$ satisfies the Slater condition if and only if the set of immobile indices is empty. Thus the emptiness of the set T^* can be considered as a constraint qualification (CQ) equivalent to the Slater-type condition for SIP.

D e f i n i t i o n 3. An immobile index $\bar{t} \in T^*$ has the order of immobility $q(\bar{t}, \bar{l})$ along a nontrivial feasible direction $\bar{l} \in L(\bar{t})$ if

1. $\left.\frac{d^i f(x,\bar{t}+\alpha\bar{l})}{d\alpha^i}\right|_{\alpha=+0} = 0, \forall x \in X, i = 0, \dots, q(\bar{t}, \bar{l})$,

2. there exists a feasible $\bar{x} \in X$ such that $\left.\frac{d^{(q(\bar{t},\bar{l})+1)} f(\bar{x},\bar{t}+\alpha\bar{l})}{d\alpha^{(q(\bar{t},\bar{l})+1)}}\right|_{\alpha=+0} \neq 0.$

Given $\bar{t} \in T$, it is easy to see that the set $L(\bar{t})$ defined in (3) is a convex polyhedral cone in $\mathbb{R}^s$. Then, according to the known results on the convex polyhedral cone's decomposition (see [3]), there exist a finite set of vectors $b_i,\ i \in \{1, \dots, p\},\ a_i, i \in I$, such that $L(\bar{t})$ admits a finite representation in the parametric form:

$$L(\bar{t}) = \{l \in \mathbb{R}^s :\ l = \sum_{i=1}^{p} \beta_i b_i + \sum_{i \in I} \alpha_i a_i,\ \alpha_i \geq 0,\ i \in I\}, \tag{4}$$

with $p = s - \text{rank}(h_k, k \in \bar{K}), |I| < \infty, \beta_i \in \mathbb{R}, i \in \{1, \dots, p\}, \alpha_i \in \mathbb{R}, i \in I$.

Vectors $b_i, i \in \{1, \dots, p\}$ satisfy the conditions $h_k^T b_i = 0,\ i = 1, ..., p$, $k \in K(\bar{t})$, and are usually referred to as *bidirectional extremal rays*. Vectors $a_i, i \in I$, in turn, satisfy the inequalities $h_k^T a_i \leq 0,\ i \in I,\ k \in K(\bar{t})$, and are called *unidirectional extremal rays*. The extremal rays can be found explicitly (see [2]).

R e m a r k 1. In the case of a pointed cone $L(\bar{t})$, the set of vectors $b_i, i = 1, \dots, p$, is empty. If $\bar{t} \in int\ T$, then the set of vectors $a_j, j \in I$, is empty and $b_i = e_i, i = 1, \dots, p = s$.

Suppose now that $\bar{t} \in T^* \subset T$ is an immobile index in (P). Consider the corresponding sets $\bar{L} = L(\bar{t})$, $\bar{K} = K_a(\bar{t})$, and suppose that the extremal rays in $\bar{L}$ are defined explicitly. Given a sufficiently small $\varepsilon > 0$, denote by $T_\varepsilon(\bar{t})$ an ε-neighborhood of $\bar{t}$ in T: $T_\varepsilon(\bar{t}) = \{t \in T :\ ||\bar{t} - t|| \leq \varepsilon\}$. From the parametric representation (4) of the cone of feasible directions $\bar{L}$, it follows that the *local constraints* $f(x,t) \leq 0, \forall t \in T_\varepsilon(\bar{t})$, can be presented in the form of the following *modified constraints*:

$$\bar{f}(x, (\beta, \alpha)) \leq 0, \forall (\beta, \alpha),\ \alpha \geq 0,\ ||(\beta, \alpha)|| \leq \varepsilon, \tag{6}$$

where $(\beta, \alpha)^T \in \mathbb{R}^{p+|I|},\ \bar{f}(x, (\beta, \alpha)) := f(x, \bar{t} + B\beta + A\alpha)$, and the columns of matrices $B \in \mathbb{R}^{s \times p}$ and $A \in \mathbb{R}^{s \times |I|}$ are presented by the bidirectional and unidirectional rays respectively.

Without loss of generality we can use here the *maximum norm* given as $||y|| = \max_{i=1,\dots,n} |y_i|$ for $y \in \mathbb{R}^n$. Then the modified constraints (6) can be considered as the box constraints w.r.t. variables (α, β), $\alpha \in \mathbb{R}^{|I|}$, $\beta \in \mathbb{R}^p$.

From the definition of the immobility index $\bar{t}$, it follows that for any $x \in X$, the vector $\bar{t}$ maximizes the function $f(x, t)$, or equivalently, vector $(\bar{\beta} = 0, \bar{\alpha} = 0)$ is a solution of a so called *lower level problem*:

$$\max_{(\beta,\alpha)} \bar{f}(x, (\beta, \alpha)), \quad \text{s.t. } \alpha \geq 0. \tag{8}$$

The first and the second order optimality conditions for the vector $(\bar{\beta} = 0, \bar{\alpha} = 0)$ in problem (8) can be formulated as follows:

$$\frac{\partial^T f(x, \bar{t})}{\partial t} b_i = 0, \; i = 1, \dots, p; \quad \frac{\partial^T f(x, \bar{t})}{\partial t} a_i \leq 0, \; i \in I, \quad \forall x \in X, \tag{9}$$

$$(\beta^T, \alpha^T)(B, A)^T \frac{\partial^2 f(x, \bar{t})}{\partial t^2} (B, A) \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \leq 0, \tag{10}$$

for all $(\beta^T, \alpha^T) \in \mathbb{R}^{p+|I|}$ such that $\alpha_i = 0$ if $\frac{\partial f^T(x,\bar{t})}{\partial t} a_i < 0$, and $\alpha_i \geq 0$ if $\frac{\partial f^T(x,\bar{t})}{\partial t} a_i = 0, i \in I$.

A s s u m p t i o n 1. Suppose that $X \neq \emptyset$, the set T is bounded and $q(l, t) \leq 1, \; \forall l \in L(t) \setminus \{0\}, \; \forall t \in T^*$.

It can be showed that Assumption 1 implies the finiteness of the set of immobile indices: $T^* = \{t^*_j, j \in J_*\}$ with $|J_*| < \infty$, and the existence of $\bar{x} \in X$ such that $|T_a(\bar{x})| < \infty$.

Suppose that the set of immobile indices and their immobility orders along the corresponding extremal rays are known ([5]). Denote:

$$I_* := \{i \in I : q(\bar{t}, a_i) = 0\} = \{i \in I : \exists x^{(i)} \in X : \frac{\partial^T f(x^{(i)}, \bar{t})}{\partial t} a_i < 0\},$$

$$I_0 := I \backslash I_* = \{i \in I : \; \frac{\partial^T f(x, \bar{t})}{\partial t} a_i = 0, \quad \forall x \in X\}.$$

Taking into account Assumption 1, we get $q(\bar{t}, b_i) = 1, \; i = 1, \dots, p$; $q(\bar{t}, a_i) \geq 1, \; i \in I_0$; $\quad q(\bar{t}, a_i) = 0, \; i \in I_*$. Then from conditions (9),(10), we conclude that for all $x \in X$ and $(\beta, \alpha_0)^T \in \mathbb{R}^p \times \mathbb{R}^{|I_0|}_+$ it holds

$$\frac{\partial^T f(x, \bar{t})}{\partial t} b_i = 0, \, i = 1, \dots, p; \; \frac{\partial^T f(x, \bar{t})}{\partial t} a_i = 0, \, i \in I_0, \tag{11}$$

$$\frac{\partial^T f(x,\bar t)}{\partial t} a_i \le 0, i \in I_*; (\beta^T, \alpha_0^T)(B, A_0)^T \frac{\partial^2 f(x,\bar t)}{\partial t^2}(B, A_0)\begin{pmatrix}\beta\\ \alpha_0\end{pmatrix} \le 0, \tag{12}$$

where $A_0 = (a_i, i \in I_0),\ \alpha_0 = (\alpha_i, i \in I_0)$.

Taking into account that for all $\bar t \in T^*$ and any $x \in X$ the relations (11), (12) are satisfied, and repeating the considerations made in [4] for the case of the box constrained index set T, we prove the following implicit optimality criterion.

Theorem 1. *Under Assumption 1, a vector $x^0 \in X$ is optimal in the convex SIP problem (P) with polyhedral index set T, if and only if there exists a finite set of indices $\{t_j,\ j \in J_a(x^0)\} \subset T_a(x^0) \setminus T^*$ with $|J_a(x^0)| \le n$, such that x^0 is optimal in the following auxiliary problem:*

$$(P_{aux}): \qquad \min_{x \in \mathbb{R}^n} c(x)$$

$$\text{s.t.} \qquad f(x, t_j) \le 0,\ j \in J_a(x^0),$$

$$f(x, t_j^*) = 0, \frac{\partial^T f(x,t_j^*)}{\partial t} B(j) = 0, \frac{\partial^T f(x,t_j^*)}{\partial t} A_0(j) = 0,\ \frac{\partial^T f(x,t_j^*)}{\partial t} A_*(j) \le 0,$$

$$(\beta^T(j), \alpha_0^T(j))(B(j), A_0(j))^T \frac{\partial^2 f(x,t_j^*)}{\partial t^2}(B(j), A_0(j))\begin{pmatrix}\beta(j)\\ \alpha_0(j)\end{pmatrix} \le 0,$$

where $(\beta(j), \alpha_0(j))^T \in \mathbb{R}^{p(j)} \times \mathbb{R}_+^{|I_0(j)|}, j \in J_*, B(j) = (b_i(j), i = 1, ..., p(t_j^*))$, and $A_0(j) = (a_i(j), i \in I_0(t_j^*)), A_*(j) = (a_i(j), i \in I_*(t_j^*))$.

Notice that the auxiliary problem (P_{aux}) is also a SIP problem but it is more easy to study and solve than the original problem (P) since

1. the infinite constraints in (P_{aux}) are quadratic w.r.t. multidimensional indices $(\beta(j), \alpha_0(j))^T \in \mathbb{R}^{p(j)} \times \mathbb{R}_+^{|I_0(j)|}$, hence this problem can be considered as a light generalization of the common semidefinite (SDP) problem (see [1]);

2. due to Assumption 1, the constraints of (P_{aux}) satisfy the Slater type condition, i.e. there exists a vector $x \in X$ such that for all $t_j^* \in T^*, j \in J_*$, it is satisfied:

$$(\beta^T(j), \alpha_0^T(j))(B(j), A_0(j))^T \tfrac{\partial^2 f(x,t_j^*)}{\partial t^2}(B(j), A_0(j))\begin{pmatrix}\beta(j)\\ \alpha_0(j)\end{pmatrix} < 0, \quad (1)$$

$$\forall (\beta(j), \alpha_0(j))^T \in \mathbb{R}^{p(j)} \times \mathbb{R}_+^{|I_0(j)|},\ (\beta(j), \alpha_0(j))^T \ne 0;$$

3. explicit optimality conditions for SDP-type problems satisfying the Slater condition can be easy formulated and can be efficiently applied to theory and practice of SIP.

The novelty of the approach presented here consists in use of the fact that the immobile indices solve the lower level problem for **all feasible** x. The analysis of the optimality conditions for the lower level problem allows one to form a new set of constraints that should be satisfied by all $x \in X$, and to formulate new optimality conditions (implicit or explicit) for the original SIP problem (P) in the form of CQ-free optimality criterion for a special auxiliary problem (P_{aux}) that has a more simple structure. Notice that in the convex case, such new optimality conditions are more strong than the known ones for SIP (see for example, [1,6]).

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